Finite Element Solutions of Boundary-value Problems in ODEs

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Outline

- Review last week on eigenvalue problems with ordinary differential equations
- Finite element methods for boundary value problems
  - Elements and shape (basis) functions
  - Getting the numerical equations by method of weighted residuals
  - Example problem and results
  - Comparison with finite difference approach

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Review Eigenvalue Problem

- Numerical eigenvalue problems occur in ODEs when the number of boundary conditions is greater than the order of the differential equation
  - Example of this is solution for burning velocity of a laminar flame
- Basic approach is to use finite-differences and transform problem into a numerical matrix eigenvalue problem

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Review Eigenvalue Problem II

- Look at simple problem with known solution as an example
  - $d^2y/dx^2 + \lambda^2 y = 0$ with $y(0) = 0$, $y(1) = 0$ and $\int y dx = 1$
  - Have three boundary conditions and only a second order equation
  - Known solution is $y = A \sin \lambda x$ with $\lambda = n\pi$
- Use second order finite differences
  - $(y_{i+1} + y_{i-1} - 2y_i)/h^2 + \lambda^2 y_i = 0$

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Review Eigenvalue Problem III

- Have matrix eigenvalue problem with $\alpha = -\lambda^2 h^2$ as the eigenvalue

$$\begin{bmatrix} -2 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \ddots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & 0 & \cdots & -2 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ \vdots \\ y_{N-2} \\ y_{N-1} \end{bmatrix} = -\lambda^2 h^2 \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ \vdots \\ y_{N-2} \\ y_{N-1} \end{bmatrix}$$

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Review Eigenvalue Problem IV

- Solve by numerical techniques for finding matrix eigenvalues
- The accuracy of the eigenvalues depends on the grid
- Often need only one (lowest or highest)
- Can only find as many eigenvalues as there are grid nodes (not counting boundary nodes)

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Review Eigenvalue Problem V

- Comparison of numerical and exact eigenvalues for grid with 4 interior nodes (h = 0.2) shows lowest error for smallest eigenvalue

Eigenvalue		Percent
Numerical	Exact	error
3.090	3.142	1.66%
5.878	6.283	6.45%
8.090	9.425	14.16%
9.511	12.566	24.31%

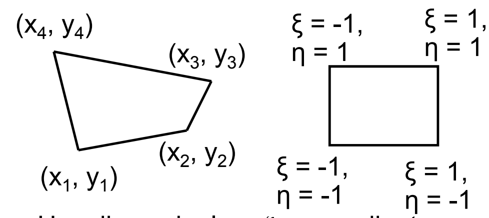
Finite Element Methods

- Designed for 2D and 3D geometries
- Can use for 1D case as example
- Basic idea is to divide region into small elements (line, area, volume)
- Use interpolating polynomial for each element
  - Represent both geometry (independent variables) and dependent variable
  - Interpolating polynomials called basis functions

Finite Element Methods II

- Analysis for individual elements is assembled into a set of nodal equations for the entire region
  - Result is set of algebraic equations for the dependent variable at nodes that are points on elements
  - Because we do not use coordinate lines for grid, it is easier to model complex engineering geometries
  - Finite-volume methods for fluid calculations

Two-dimensional Element



- Use dimensionless ξ-η coordinate system for basis functions
- Each element has several shape or basis functions, ϕ_i

Shape (Basis) Functions

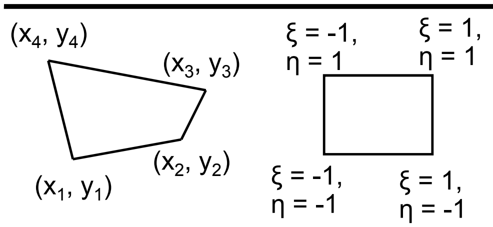
- Model geometry and dependent variable over the element
- Use of same basis functions for both is called isoparametric element
- Shape functions associated with element nodes such that $\phi_i(\mathbf{x}_{(j)}) = \delta_{ij}$

$$x = \sum_{i=1}^{4} x_i \varphi_i \quad y = \sum_{i=1}^{4} y_i \varphi_i \quad T = \sum_{i=1}^{4} T_i \varphi_i$$

Shape (Basis) Functions II

- Simplest shape functions are linear for 1D or bilinear for 2D
- For a linear element between nodes i (at $\xi = -1$) and i + 1 (at $\xi = 1$) we have $\phi_i = (1 - \xi)/2$ and $\phi_{i+1} = (1 + \xi)/2$
- $x = x_i \phi_i + x_{i+1} \phi_{i+1}$ is correct at 1D nodes
- Bilinear functions for 2D element have the form $(1 \pm \xi)(1 \pm \eta)/2$

Bilinear Shape Functions



Note: $\phi_i(\mathbf{x}_{(j)}) = \delta_{ij}$

$$\varphi_1 = \frac{(1-\xi)(1-\eta)}{4} \qquad \varphi_2 = \frac{(1+\xi)(1-\eta)}{4}$$

$$\varphi_3 = \frac{(1+\xi)(1+\eta)}{4} \qquad \varphi_4 = \frac{(1-\xi)(1+\eta)}{4}$$

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Modeling Differential Equation

- Look at same example used for finite differences: $d^2T/dx^2 + a^2T = 0$
- Equation for T in terms of basis functions gives approximate value $\hat{T} = \sum_{i=1}^{N} T_i\varphi_i$
- Seek solution in which differential equation is satisfied in an average way over the region; w_i is weighting function

$$\int_0^L w_i(x)\left[\frac{d^2\hat{T}}{dx^2} + a^2\hat{T}\right]dx = 0 \qquad i = 0,\ldots N$$

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Modeling Differential Equation II

- Last equation on previous chart called method of weighted residuals (MWR)
- Various choices are used for weighting functions, w_i
- Galerkin method uses $w_i = \phi_i$
- Known to match variational results for linear problems

$$\int_0^L \varphi_i(x)\left[\frac{d^2\hat{T}}{dx^2} + a^2\hat{T}\right]dx = 0 \qquad i = 0,\ldots N$$

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Modeling Differential Equation III

- Use integration by parts to eliminate second derivatives which give zero for linear basis functions

$$\int_0^L \varphi_i \frac{d^2\hat{T}}{dx^2}dx = \int_0^L \varphi_i \frac{d}{\cancel{dx}}\left(\frac{d\hat{T}}{dx}\right)\cancel{dx} = \int_0^L \underbrace{\varphi_i}_{u}\, \underbrace{d\left(\frac{d\hat{T}}{dx}\right)}_{dv}$$

$$= \left[\varphi_i \frac{d\hat{T}}{dx}\right]_0^L - \int_0^L \frac{d\hat{T}}{dx}d\varphi_i = \left[\varphi_i \frac{d\hat{T}}{dx}\right]_0^L - \int_0^L \frac{d\hat{T}}{dx}\frac{d\varphi_i}{dx}dx$$

Multiply by dx/dx

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Modeling Differential Equation IV

- Substituting result of integration by parts into original equation gives (for each i)

$$\boxed{\int_0^L \varphi_i\left[\frac{d^2\hat{T}}{dx^2} + a^2\hat{T}\right]dx} = \left[\varphi_i \frac{d\hat{T}}{dx}\right]_0^L - \int_0^L \frac{d\hat{T}}{dx}\frac{d\varphi_i}{dx}dx$$

$$+\int_0^L \varphi_i a^2\hat{T}\,dx = \boxed{\left[\varphi_i \frac{d\hat{T}}{dx}\right]_0^L - \int_0^L \left[\frac{d\hat{T}}{dx}\frac{d\varphi_i}{dx} - \varphi_i a^2\hat{T}\right]dx = 0}$$

- Next steps: handle boundary terms, separate into elements, and introduce shape functions for T

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Modeling Differential Equation V

- At x = 0, $\phi_0 = 1$ and all other $\phi_i = 0$
- At x = L, $\phi_N = 1$ and all other $\phi_i = 0$
- Have different equations for i = 0 and i = N

$$i = 0 \qquad -\left.\frac{d\hat{T}}{dx}\right|_{x=0} - \int_0^L \left[\frac{d\hat{T}}{dx}\frac{d\varphi_0}{dx} - \varphi_0 a^2\hat{T}\right]dx = 0$$

$$i = N \qquad \left.\frac{d\hat{T}}{dx}\right|_{x=L} - \int_0^L \left[\frac{d\hat{T}}{dx}\frac{d\varphi_N}{dx} - \varphi_N a^2\hat{T}\right]dx = 0$$

$$\text{All other } i \qquad \int_0^L \left[\frac{d\hat{T}}{dx}\frac{d\varphi_i}{dx} - \varphi_i a^2\hat{T}\right]dx = 0$$

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Modeling Differential Equation VI

- Integrals in summation are evaluated over each element from x_i to x_{i+1}
- Substitute basis function equation for T into these integrals $\hat{T} = \sum_{j=0}^{N} T_j \varphi_j$

$$\int_0^L \frac{d\hat{T}}{dx}\frac{d\varphi_i}{dx}dx = \int_0^L \frac{d\sum_{j=0}^{N} T_j\varphi_j}{dx}\frac{d\varphi_i}{dx}dx = \sum_{j=0}^{N} T_j \int_0^L \frac{d\varphi_j}{dx}\frac{d\varphi_i}{dx}dx$$

$$\int_0^L \varphi_i a^2 \hat{T} dx = \int_0^L \left[\varphi_i a^2 \sum_{j=0}^{N} T_j \varphi_j\right] dx = \sum_{j=0}^{N} T_j \int_0^L \varphi_i a^2 \varphi_j dx$$

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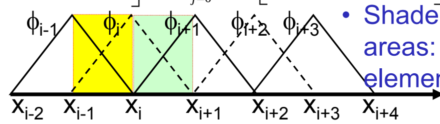
Modeling Differential Equation VI

- Look shape functions for each ϕ_i

$$\int_0^L \left[\frac{d\hat{T}}{dx}\frac{d\varphi_i}{dx} - \varphi_i a^2 \hat{T}\right]dx = \sum_{j=0}^{N} T_j \int_0^L \left[\frac{d\varphi_j}{dx}\frac{d\varphi_i}{dx} - \varphi_i a^2 \varphi_j\right]dx = 0$$



- Shaded areas: single elements

- All basis functions except ϕ_{i-1} and ϕ_{i+1} are zero where ϕ_i is not zero ($x_{i-1} < x < x_{i+1}$)

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Modeling Differential Equation VI

- Have only three basis function pairs to consider in any integral for ϕ_i: i and i –1, i and i, i and i +1

$$A_{i,i-1} = \int_{x_{i-1}}^{x_{i+1}} \left[\frac{d\varphi_{i-1}}{dx}\frac{d\varphi_i}{dx} - \varphi_i a^2 \varphi_{i-1}\right]dx \qquad A_{i,i} = \int_{x_{i-1}}^{x_{i+1}} \left[\frac{d\varphi_i}{dx}\frac{d\varphi_i}{dx} - \varphi_i a^2 \varphi_i\right]dx \qquad A_{i,i+1} = \int_{x_{i-1}}^{x_{i+1}} \left[\frac{d\varphi_{i+1}}{dx}\frac{d\varphi_{i1}}{dx} - \varphi_i a^2 \varphi_{i+1}\right]dx$$

$$\sum_{j=0}^{N} T_j \int_0^L \left[\frac{d\varphi_j}{dx}\frac{d\varphi_i}{dx} - \varphi_i a^2 \varphi_j\right]dx = A_{i,i-1}T_{i-1} + A_{i,i}T_i + A_{i,i+1}T_{i+1} = 0$$

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Linear Basis Functions

$$\varphi_i(x) = \begin{cases} 0 & x \le x_{i-1} \\ \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & x_{i-1} \le x \le x_i \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i} & x_i \le x \le x_{i+1} \\ 0 & x \ge x_{i+1} \end{cases}$$

Get needed basis functions by substituting i-1 and i +1 for i

Substitute basis functions and derivatives into integrals

$$\frac{d\varphi_i(x)}{dx} = \begin{cases} 0 & x \le x_{i-1} \\ \dfrac{1}{x_i - x_{i-1}} & x_{i-1} \le x \le x_i \\ \dfrac{-1}{x_{i+1} - x_i} & x_i \le x \le x_{i+1} \\ 0 & x \ge x_{i+1} \end{cases}$$

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Results of Integration

$$\alpha_i = \frac{a^2}{3}(x_{i+1} - x_i) - \frac{1}{x_{i+1} - x_i} \qquad \beta_i = \frac{a^2}{6}(x_{i+1} - x_i) + \frac{1}{x_{i+1} - x_i}$$

$$\sum_{j=0}^{N} T_j \int_0^L \left[\frac{d\varphi_j}{dx}\frac{d\varphi_i}{dx} - \varphi_i a^2 \varphi_j\right]dx = A_{i,i-1}T_{i-1} + A_{i,i}T_i + A_{i,i+1}T_{i+1} = 0$$

$$A_{i,i-1} = \int_{x_{i-1}}^{x_{i+1}} \left[\frac{d\varphi_{i-1}}{dx}\frac{d\varphi_i}{dx} - \varphi_i a^2 \varphi_{i-1}\right]dx = \beta_{i-1}$$

$$A_{i,i} = \int_{x_{i-1}}^{x_{i+1}} \left[\frac{d\varphi_i}{dx}\frac{d\varphi_i}{dx} - \varphi_i a^2 \varphi_i\right]dx = \alpha_i + \alpha_{i-1}$$

$$A_{i,i+1} = \int_{x_{i-1}}^{x_{i+1}} \left[\frac{d\varphi_{i+1}}{dx}\frac{d\varphi_{i1}}{dx} - \varphi_i a^2 \varphi_{i+1}\right]dx = \beta_i$$

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Constant Steps $x_{i+1} – x_i = h$

$$\alpha_i = \frac{ha^2}{3} - \frac{1}{h} \qquad \beta_i = \frac{ha^2}{6} + \frac{1}{h}$$

$$\sum_{j=0}^{N} T_j \int_0^L \left[\frac{d\varphi_j}{dx}\frac{d\varphi_i}{dx} - \varphi_i a^2 \varphi_j\right]dx = A_{i,i-1}T_{i-1} + A_{i,i}T_i + A_{i,i+1}T_{i+1} = 0$$

$$A_{i,i-1} = \int_{x_{i-1}}^{x_{i+1}} \left[\frac{d\varphi_{i-1}}{dx}\frac{d\varphi_i}{dx} - \varphi_i a^2 \varphi_{i-1}\right]dx = \frac{ha^2}{6} + \frac{1}{h}$$

$$A_{i,i} = \int_{x_{i-1}}^{x_{i+1}} \left[\frac{d\varphi_i}{dx}\frac{d\varphi_i}{dx} - \varphi_i a^2 \varphi_i\right]dx = \frac{2ha^2}{3} - \frac{2}{h}$$

$$A_{i,i+1} = \int_{x_{i-1}}^{x_{i+1}} \left[\frac{d\varphi_{i+1}}{dx}\frac{d\varphi_{i1}}{dx} - \varphi_i a^2 \varphi_{i+1}\right]dx = \frac{ha^2}{6} + \frac{1}{h}$$

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Equations to be Solved

$$\alpha_0 T_0 + \beta_0 T_1 = -\left.\frac{dT}{dx}\right|_{x=x_0} \qquad \alpha_{N-1} T_{N-1} + \beta_{N-1} T_N = \left.\frac{dT}{dx}\right|_{x=x_N}$$

$$\beta_i T_{i-1} + (\alpha_i + \alpha_{i+1}) T_i + \beta_{i+1} T_{i+1} = 0 \quad i = 1, \ldots, N-1$$

- Tridiagonal system of N+1 equations with N+3 variables
  - N+1 temperature values and 2 boundary gradients
  - Boundary conditions will specify two other equations

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Boundary Gradients

- If we have Dirichlet boundary conditions, we can solve for temperatures then find gradients
- For Neumann or mixed boundary conditions, we must include gradients in tridiagonal solution
- Write boundary conditions as a dT/dx + bT = c and make $g_0 = dT/dx|_{x=0}$ the first variable and $g_L = dT/dx|_{x=L}$ the last one

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Finite Element Equations

- Equations below only handle boundary conditions with specified gradients ($a_0 \neq 0$)

$$\begin{bmatrix} a_0 & b_0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ -1 & \alpha_0 & \beta_0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & \beta_1 & \alpha_1+\alpha_2 & \beta_2 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \beta_2 & \alpha_2+\alpha_3 & \beta_3 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 0 & \beta_3 & \ddots & & & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \cdots & \ddots & \beta_{N-1} & 0 \\ 0 & 0 & 0 & 0 & \cdots & \cdots & \beta_{N-1} & \alpha_{N-1}+\alpha_N & -1 \\ 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & b_N & a_N \end{bmatrix} \begin{bmatrix} g_0 \\ T_0 \\ T_1 \\ T_2 \\ \vdots \\ \vdots \\ T_{N-1} \\ T_N \\ g_N \end{bmatrix} = \begin{bmatrix} c_0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 0 \\ c_N \end{bmatrix}$$

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Solution Errors for a = 2

N	100	100	10	10
Method	FD	FE	FD	FE
e_{RMS}	$1.7\text{x}10^{-5}$	$1.7\text{x}10^{-5}$	$1.8\text{x}10^{-3}$	$1.8\text{x}10^{-3}$
e_{max}	$2.4\text{x}10^{-5}$	$2.4\text{x}10^{-5}$	$2.4\text{x}10^{-3}$	$2.4\text{x}10^{-3}$
$e_{grad}(0)$	$3.6\text{x}10^{-4}$	$7.0\text{x}10^{-5}$	$3.6\text{x}10^{-2}$	$7.0\text{x}10^{-3}$
$e_{grad}(L)$	$2.1\text{x}10^{-4}$	$9.6\text{x}10^{-5}$	$1.8\text{x}10^{-2}$	$9.5\text{x}10^{-3}$

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Solution Errors for a = 0.2

N	100	100	10	10
Method	FD	FE	FD	FE
e_{RMS}	$6.2\text{x}10^{-10}$	$6.2\text{x}10^{-10}$	$6.5\text{x}10^{-8}$	$6.5\text{x}10^{-8}$
e_{max}	$8.6\text{x}10^{-10}$	$8.6\text{x}10^{-10}$	$8.5\text{x}10^{-8}$	$8.5\text{x}10^{-8}$
$e_{grad}(0)$	$1.3\text{x}10^{-6}$	$2.2\text{x}10^{-9}$	$1.3\text{x}10^{-4}$	$2.2\text{x}10^{-7}$
$e_{grad}(L)$	$1.3\text{x}10^{-6}$	$4.5\text{x}10^{-9}$	$1.3\text{x}10^{-4}$	$4.4\text{x}10^{-7}$

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Notes on the Error

- The formulations used here for finite elements and finite differences have second order error
  - Notes both equations almost the same
- Although temperature errors are similar, finite elements gives smaller errors in the gradients
- The heat source parameter, $a^2 = b/k$, can change the error for a given h

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Finite Element Grids

- Elements allow fitting complex objects used in almost all engineering designs
- Modern engineering software usually has grid generation that allows users to specify overall data on grid sizes and then has a program that generates the finite-element grid
- Element quality is a prime concern when considering the grid generated

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Grid (Mesh) Quality

- Finite element mesh quality
- Grid generation programs for finite-element analysis of engineering problems report measures of grid quality
  - Skewness
  - Smoothness
  - Aspect ratio

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Mesh Quality: Skewness

- Based on difference from an equilateral element
- Use quadrilateral elements as an example: equilateral elements have 90-degree angles
- Skewness = Max$\left[\frac{\theta_{max}-90}{90}, \frac{90-\theta_{min}}{90}\right]$
- Best value is zero; worst value is 1

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Other Mesh Quality Issues

- Resolution – mesh should be finer in areas where there are significant changes such as fluid boundary layers, and stress concentrations
- Smoothness – changes in element sizes should be gradual
- Cell aspect ratios should usually not deviate more than 20% from uniform shaped cells except in special cases

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